Chapter 1: Functions

Our Number System

The integers: $\mathbb{Z}$ counting numbers together with their negatives:
$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

The real numbers: $\mathbb{R}$ are all of the possible decimal numbers between the integers. Divide this into two categories:

a) Rational numbers $\mathbb{Q}$, decimals which terminate or repeat:
$4, 1.5, 1.3333\ldots, 2.091091091\ldots$

b) Irrational numbers $\mathbb{I}$, decimals which repeat without pattern:
$3.1415926\ldots, \sqrt{2}, e$

Algebraic Properties of $\mathbb{R}$:

1. $x + 0 = x$
2. $x - x = 0$
3. $x + y = y + x$
4. $(x + y) + z = x + (y + z)$
5. $1 \times x = x$
6. $x \times \frac{1}{x} = 1$
7. $x \times y = y \times x$ we usually write multiplication by concatenation: $xy = yx$
8. $x(yz) = (xy)z$
9. $x(y + z) = xy + xz$

Order Properties of $\mathbb{R}$: less than/greater than

1. Every pair of numbers is comparable: $x \leq y$ or $x \geq y$
2. If $x \leq y$, then $x + z \leq y + z$
3. If $x \leq y$, then $xz \leq yz$ if $z \geq 0$ or $xz \geq yz$ if $z \leq 0$
4. If $x > 0$ and $y > 0$ then $xy > 0$ and $x + y > 0$

Defintion. For a set of numbers S, a number M is an *upper-bound* of S if $M > x$ for all members x in S.
A number m is a *lower-bound* of S if $m < x$ for all members x in S.

Defintion. For a set of numbers S, a number M is a *least upper-bound* of S if it is the smallest of all upper-bounds of S
A number m is a *greatest lower-bound* of S if it is the largest of all lower-bounds of S.

Example. The interval $I = (0, 1)$ is the set of numbers $0 < x < 1$.

1. Find an upper-bound of I.

2. Find a lower-bound of I.

3. What is the l.u.b. of I? The g.l.b of I?

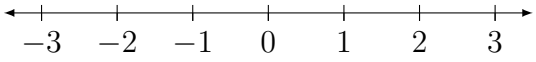
Order Properties Continued

5. For any two numbers x and y there is a number between them.

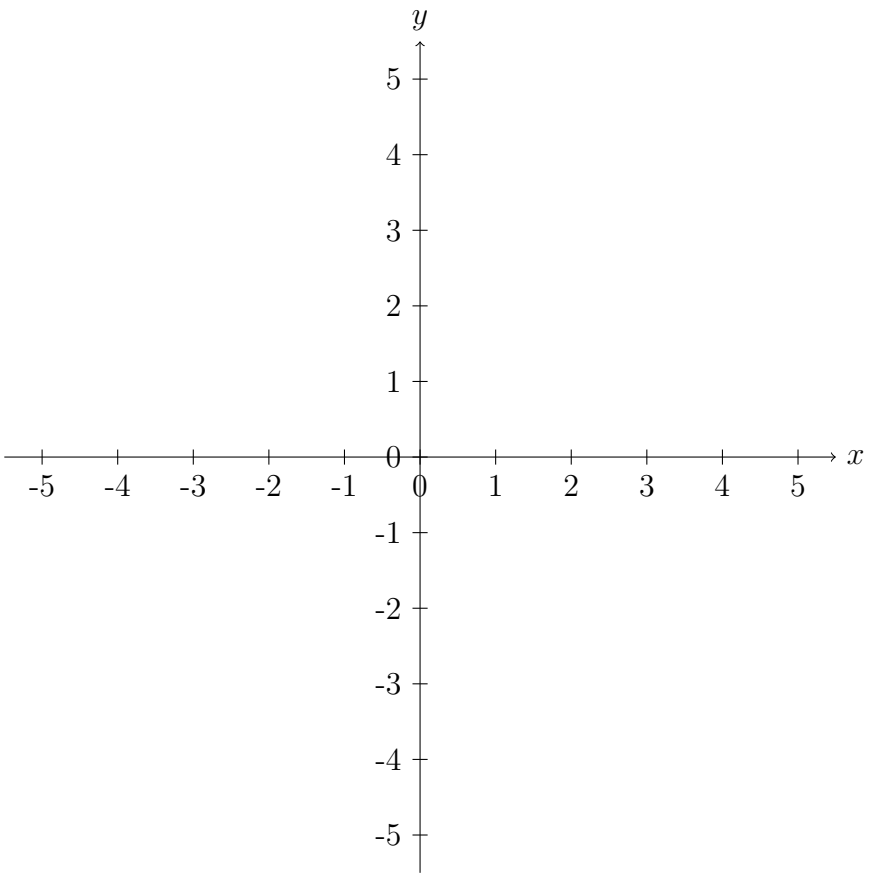
6. Any set of numbers with an upper-bound has a least upper-bound.

7. Any set of numbers with a lower-bound has a greatest lower-bound.

Real numbers fit in order on a number line:



And we plot ordered pairs (x, y) on the Cartesian plane:



Functions

These are the "objects" that we study in calculus!

Defintion (Functions)**.** A *function* f is a rule that assigns to each input x an output $f(x)$. The set of all valid inputs is called the *domain*. The set of all possible outputs is called the *range*.
We can write a function with the symbol

$$f : D \to \mathbb{R}$$

Example. Find the domain of:

1. $f(x) = x^2$

2. $g(x) = \sqrt{x-2}$

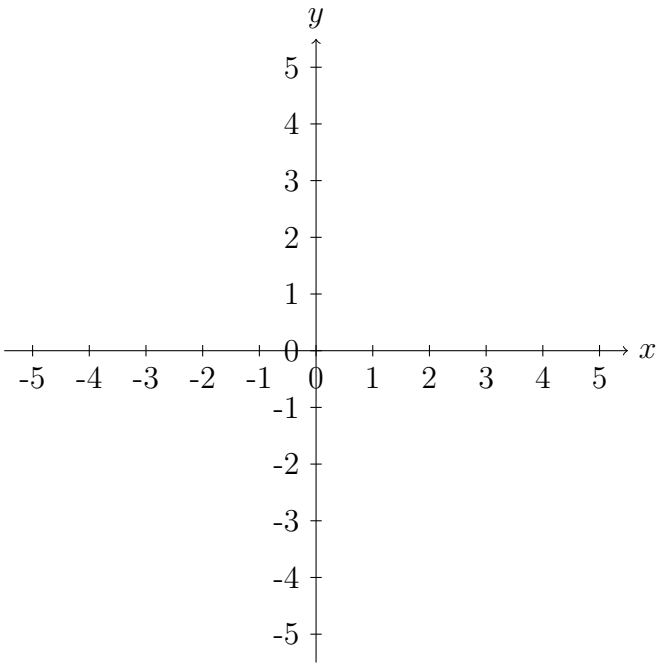
3. $h(x) = \dfrac{1}{x+3}$

Function rules can be particularly hard to understand

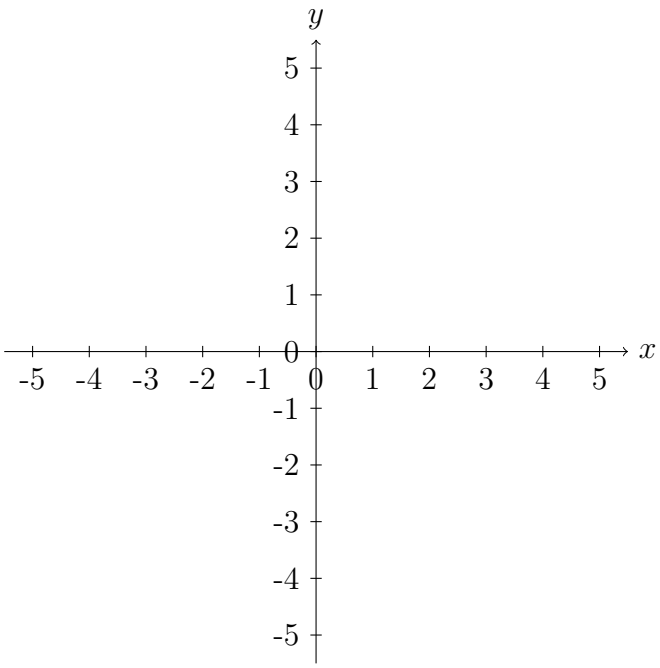
$$f(x) = \int_0^x t^x e^{-t}\, dt$$

The *graph* of a function is the set of all points $(x, f(x))$ in the plane.

Example. Graph the function $f(x) = x^2 - 2x$



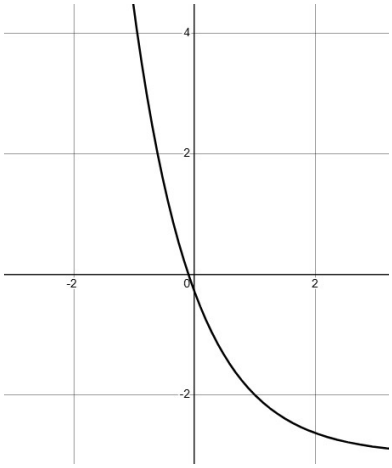
A graph represents a function if it passes the vertical line test

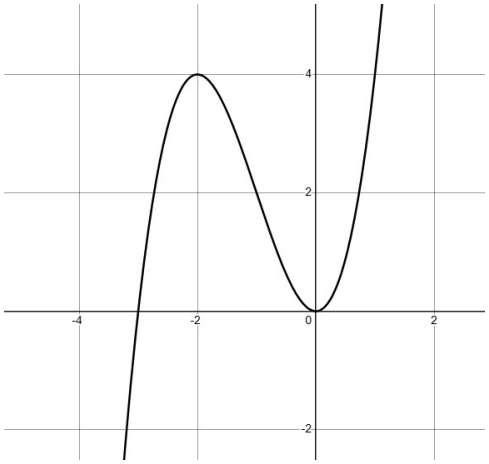


Defintion (Composite Functions)**.** For two function f and g we can create the *composition* $f \circ g$ defined by $(f \circ g)(x) = f(g(x))$. This is evaluated in two steps: $y = f(u)$ where $u = g(x)$.

Defintion (Inverse Function)**.** Given a function f, its *inverse* (if it exists) is a function f^{-1} such that if $y = f(x)$ then $f^{-1}(y) = x$.

A function has an inverse only if it passes the horizontal line test. We call these functions one-to-one.





We can restrict domain to obtain invertible function

Example. Find the inverse of $f(x) = 2x + 6$ and $g(x) = \sqrt{x^3 - 1}$.

Example. Let $f(x) = x^2 - 9$ and $g(x) = x^3$. Find

1. $f \circ g$

2. $g(1/u)$

3. $g(\sqrt{x+4})$

Example. Decompose the following functions:

1. $f(x) = (x^3 - 4)^5$

2. $g(x) = \dfrac{1}{\sqrt{x^3 - 1}}$

Types of Functions

The must important functions for us to understand starting out are linear functions and graphs of lines.

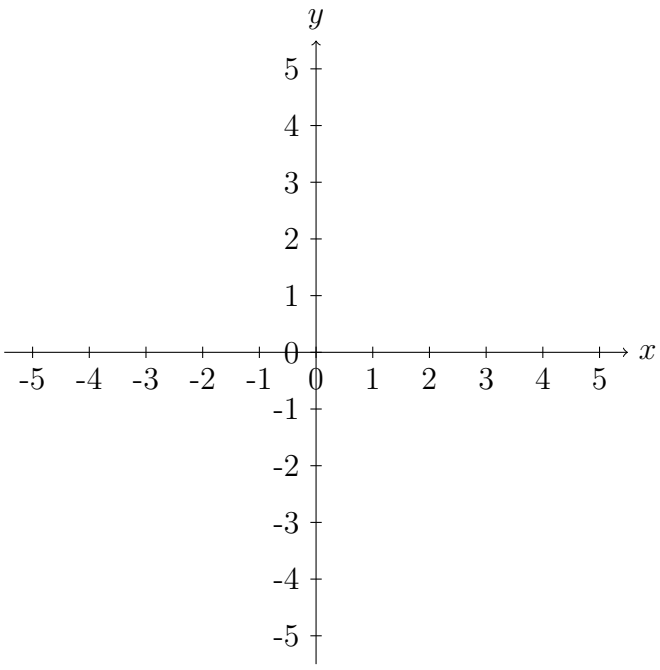
A line is described by two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$.

The *slope* of the line through these points is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The line through P, Q can be described by any of the equations

The slope is a fixed parameter for a given line!



Elementary Functions

- *Polynomials* are functions of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

- *Rational functions* are of the form

$$\frac{p(x)}{q(x)}$$

  where p and q are polynomials.

- *Algebraic functions* perform algebraic operations on polynomials and rational functions. Like:

$$f(x) = \sqrt{x^2 - 3x}$$

- *Exponential functions* are of the form $f(x) = b^x$ where $b > 0$. Their inverse functions are called *logarithmic functions*

- *Trigonometric functions* are the familiar $sin, cos, tan,$ etc.

Some functions have symmetries that are useful in calculus

Let f be a function

- f is *even* if $f(-x) = f(x)$ for all x in the domain.
- f is *odd* if $f(-x) = -x$ for all x in the domain.
- f has a *period* of c if $f(x) = f(x + c)$ for all x in the domain and c is the smallest such number for which this property holds.